

# The Fick-Jacobs diffusion equation as a Schrödinger equation

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## Abstract

It is shown a relation between the Fick-Jacobs equation and Schrödinger equation. The Fick-Jacobs equation describes the diffusion in a channel with a shape of a surface of revolution. For the case of constant diffusion coefficient the following results are obtained: the case of the conical channel is mapped to quantum free particle; if the shape of channel is a throat is obtained a relation with quantum particle in a constant potential; also the sinusoidal channel is related with quantum particle in a constant potential; in addition, when the channel cross section varies as a quadratic exponential, is shown its equivalence with the quantum harmonic oscillator. The general case of variable diffusion coefficient is also considered. In this case a change of variable is proposed, and if it is invertible, the Fick-Jacobs equation is equivalent to the Schrödinger equation of a particle in an effective potential.

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# 1 Introduction

Recently, mathematical techniques used to study certain physical systems have been used to address another physical systems that are completely different. For example, with the so called *AdS/CFT* correspondence, using theories of gravity it has been able to find results in field theories [1]. Moreover, with this duality it has been possible to relate condensed matter systems with gravitational systems [2, 3]. In addition, Unruh found an analogy between the theory of fluids and some spacetimes like black holes [4], other works on this subject can be seen in [5]. Is worth stressing that by studying some metamaterials it can be simulated certain spacetimes [6]. Also, recently methods of physics haven been used to address some problems in other disciplines such as finance and economics. For instance, the Black-Scholes equation that plays an important role in finance, can be mapped to the Schrödinger equation of a free nonrelativistic particle [7]. Is worth noticing that the Hamiltonian obtained in such case has a supersymmetric partner [8]. With such Hamiltonians one can propose a generalization of supersymmetric quantum mechanics [9].

Now, there are many problems related to the transport of particles through channels or other confined system. The most simple way to study them is through the Fick diffusion equation [14], that can be mapped to Schrödinger equation. However in several cases one need to use more general equations to describe the problem [10, 11]. Particularly, when the channel has the shape of a surface of revolution whose cross section has area  $A(x)$ , the problem reduces to solving an effective diffusion equation in one dimension known as the Fick-Jacobs equation [12, 13]. This is an effective one dimensional equation that approximates the diffusion on higher dimensions for confined systems by introducing an effective diffusion coefficient that varies with position  $D(x)$ :

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[ D(x)A(x) \frac{\partial}{\partial x} \left( \frac{C(x,t)}{A(x)} \right) \right], \quad (1)$$

where  $C(x,t)$  is the concentration of particles that diffuse. Notice that if the area and the diffusion coefficient are constant, we obtain the usual Fick equation. In standard references of Fick-Jacobs diffusion [10, 11, 13] one of their main goals is to find the corresponding relaxation time of the system. This is achieved either by analytical and perturbative methods or through

numerical simulations.

In this work is shown a relation between the Fick-Jacobs diffusion equation and the Schrödinger equation. It is shown that such relation can be realized in several cases. When the diffusion coefficient is constant the following results are obtained: the case of the conical channel is mapped to quantum free particle; if the shape of channel is a throat, we have a relation with quantum particle in a constant potential; also the sinusoidal channel is related with quantum particle in a constant potential; in addition when the channel cross section varies as a quadratic exponential, we obtain a relation with the quantum harmonic oscillator. For the general case when the diffusion coefficient is not a constant but a function of the position  $D(x)$ , it is found an equivalence between Fick-Jacobs and Schrödinger equations, if a certain change of variable is invertible. The result is that the Fick-Jacobs equation with variable diffusion coefficient can be mapped to the Schrödinger equation for a nonrelativistic particle in an effective potential.

This paper is organized as follows: In section 2 is presented a review of quantum mechanics with a generalized momentum operator. In section 3 is studied the Fick-Jacobs equation for the conical channel. In section 4 is considered the general case with constant diffusion coefficient and are also presented three cases with exact solution. In section 5 is analyzed the general case. Finally in section 6 the summary is given.

## **2 Generalized quantum mechanical momentum operator**

In this section we introduce a generalized momentum operator with which one can define more than one single Hamiltonian; in particular, a pair of supersymmetric Hamiltonians and two non-Hermitian Hamiltonians. With the non-Hermitian Hamiltonians we can obtain wave equations that are equivalent to Schrödinger equations. This section describes the method used later to demonstrate that the Fick-Jacobs equation is equivalent to a Schrödinger equation.

Lets consider the following Hermitian operator

$$\hat{P} = -i \frac{\partial}{\partial x}, \quad (2)$$

that is the usual definition of momentum operator in quantum mechanics with  $\hbar = 1$ . We can define a different non-Hermitian operator by adding the derivative of some arbitrary function  $f(x)$  to the original momentum giving:

$$\hat{P}_{(f)} = e^{f(x)} \hat{P} e^{-f(x)} = \hat{P} + i \frac{\partial f(x)}{\partial x}. \quad (3)$$

This generalized momentum operator was first studied by P.M.A Dirac [15].

With the new momentum operator (3), we can build the following two Hermitian Hamiltonians:

$$\hat{H}_1 = \alpha^2 \hat{P}_{(f)}^\dagger \hat{P}_{(f)} = \alpha^2 \left( \hat{P}^2 + \frac{d^2 f}{dx^2} + \left( \frac{df}{dx} \right)^2 \right), \quad (4)$$

$$\hat{H}_2 = \alpha^2 \hat{P}_{(f)} \hat{P}_{(f)}^\dagger = \alpha^2 \left( \hat{P}^2 - \frac{d^2 f}{dx^2} + \left( \frac{df}{dx} \right)^2 \right), \quad (5)$$

where  $\alpha$  is a constant. Notice that if we substitute in (4) and (5) the following definition

$$f(x) = \int_0^x W(u) du, \quad (6)$$

we obtain

$$\hat{H}_1 = \alpha^2 \left( \hat{P}^2 + \frac{dW}{dx} + W^2 \right), \quad (7)$$

$$\hat{H}_2 = \alpha^2 \left( \hat{P}^2 - \frac{dW}{dx} + W^2 \right). \quad (8)$$

With these two Hamiltonians one can construct a supersymmetric quantum mechanics [16]. Furthermore, with the generalized momentum operator (3) we can also build two non-Hermitian Hamiltonians:

$$\hat{H}_3 = \beta^2 \hat{P}_{(f)}^\dagger \hat{P}_{(f)}^\dagger = \beta^2 \left( \hat{P}^2 - 2i \frac{df}{dx} \hat{P} - \frac{d^2 f}{dx^2} - \left( \frac{df}{dx} \right)^2 \right), \quad (9)$$

$$\hat{H}_4 = \beta^2 \hat{P}_{(f)} \hat{P}_{(f)} = \beta^2 \left( \hat{P}^2 + 2i \frac{df}{dx} \hat{P} + \frac{d^2 f}{dx^2} - \left( \frac{df}{dx} \right)^2 \right), \quad (10)$$

where  $\beta$  is a constant.

Lets notice that the transformation  $f(x) \rightarrow -f(x)$ , implies that  $\hat{H}_3 \rightarrow \hat{H}_4$ , which means that once we solve one problem, we will have the solution of the other. It should be noticed that this kind of Hamiltonians arise in the so-called quantum finance [7], these Hamiltonians allow to generalize the supersymmetric quantum mechanics [9]. A generalized quantum mechanics with non-Hermitian operators is studied in [17].

With the Hamiltonian  $\hat{H}_3$  and  $\hat{H}_4$  we can write the following wave equations

$$i \frac{\partial \psi_3(x, t)}{\partial t} = \hat{H}_3 \psi_3(x, t), \quad (11)$$

$$i \frac{\partial \psi_4(x, t)}{\partial t} = \hat{H}_4 \psi_4(x, t), \quad (12)$$

where, as usual  $\psi_j$ , with  $j = 3, 4$ , is the wave function. The Hamiltonians (9) and (10) are non-Hermitian, but the wave equations (11) and (12) are equivalent to a nonrelativistic free particle. To prove this claim first note that

$$\begin{aligned} \hat{P}_{(f)}^\dagger \hat{P}_{(f)}^\dagger &= e^{-f(x)} \hat{P} e^{f(x)} e^{-f(x)} \hat{P} e^{f(x)} = e^{-f(x)} \hat{P}^2 e^{f(x)}, \\ \hat{P}_{(f)} \hat{P}_{(f)} &= e^{f(x)} \hat{P} e^{-f(x)} e^{f(x)} \hat{P} e^{-f(x)} = e^{f(x)} \hat{P}^2 e^{-f(x)}, \end{aligned} \quad (13)$$

which corresponds to the equations

$$\begin{aligned} i \frac{\partial \psi_3(x, t)}{\partial t} &= \hat{H}_3 \psi_3(x, t) = \beta^2 e^{-f(x)} \hat{P}^2 e^{f(x)} \psi_3(x, t), \\ i \frac{\partial \psi_4(x, t)}{\partial t} &= \hat{H}_4 \psi_4(x, t) = \beta^2 e^{f(x)} \hat{P}^2 e^{-f(x)} \psi_4(x, t), \end{aligned}$$

from where we have

$$\begin{aligned} i \frac{\partial (e^{f(x)} \psi_3(x, t))}{\partial t} &= \beta^2 \hat{P}^2 (e^{f(x)} \psi_3(x, t)), \\ i \frac{\partial (e^{-f(x)} \psi_4(x, t))}{\partial t} &= \beta^2 \hat{P}^2 (e^{-f(x)} \psi_4(x, t)), \end{aligned}$$

both equations represent the wave equation of a free particle with corresponding wave function  $e^{\pm f}\psi_j$ . The general solutions of these equations are:

$$\begin{aligned} e^{f(x)}\psi_3(x, t) &= e^{-i\beta^2\hat{P}^2t} \left( e^{f(x)}\psi_{03}(x) \right), \\ e^{-f(x)}\psi_4(x, t) &= e^{-i\beta^2\hat{P}^2t} \left( e^{-f(x)}\psi_{04}(x) \right), \end{aligned} \quad (14)$$

where  $\psi_3(x, 0) = \psi_{03}(x)$ ,  $\psi_4(x, 0) = \psi_{04}(x)$ , are the initial conditions. Thus, the general solutions of the equations (11) and (12) are given by:

$$\begin{aligned} \psi_3(x, t) &= \left( e^{-f(x)} e^{-i\beta^2\hat{P}^2t} e^{f(x)} \right) \psi_{03}(x), \\ \psi_4(x, t) &= \left( e^{f(x)} e^{-i\beta^2\hat{P}^2t} e^{-f(x)} \right) \psi_{04}(x). \end{aligned} \quad (15)$$

Similarly, one can prove that the wave equations with a potential  $V(x)$

$$i \frac{\partial \psi_3(x, t)}{\partial t} = \left( \beta^2 \hat{P}_{(f)}^\dagger \hat{P}_{(f)} + V(x) \right) \psi_3(x, t) \quad (16)$$

$$i \frac{\partial \psi_4(x, t)}{\partial t} = \left( \beta^2 \hat{P}_{(f)} \hat{P}_{(f)} + V(x) \right) \psi_4(x, t), \quad (17)$$

are equivalent to the following Schrödinger equations

$$i \frac{\partial \left( e^{f(x)} \psi_3(x, t) \right)}{\partial t} = \hat{H} \left( e^{f(x)} \psi_3(x, t) \right), \quad (18)$$

$$i \frac{\partial \left( e^{-f(x)} \psi_4(x, t) \right)}{\partial t} = \hat{H} \left( e^{-f(x)} \psi_4(x, t) \right), \quad (19)$$

where the Hamiltonian is just

$$\hat{H} = \beta^2 \hat{P}^2 + V(x). \quad (20)$$

The general solutions of the equations (18) and (19) are respectively:

$$e^{f(x)}\psi_3(x, t) = e^{-i\hat{H}t} \left( e^{f(x)}\psi_{03}(x) \right), \quad (21)$$

$$e^{-f(x)}\psi_4(x, t) = e^{-i\hat{H}t} \left( e^{-f(x)}\psi_{04}(x) \right), \quad (22)$$

therefore, the general solutions of the equations (16) and (17) are given by

$$\psi_3(x, t) = \left( e^{-f(x)} e^{-i\hat{H}t} e^{f(x)} \right) \psi_{03}(x), \quad (23)$$

$$\psi_4(x, t) = \left( e^{f(x)} e^{-i\hat{H}t} e^{-f(x)} \right) \psi_{04}(x), \quad (24)$$

where the initial conditions are  $\psi_3(x, 0) = \psi_{03}(x)$ ,  $\psi_4(x, 0) = \psi_{04}(x)$ .

In the next section we will use these results to show that the Fick-Jacobs equation is equivalent to a Schrödinger equation.

### 3 Fick-Jacobs equation for a conical channel

In this section we show that the Fick-Jacobs equation for the case where the diffusion coefficient is constant and the channel has a conical shape, is equivalent to a Schrödinger equation of a nonrelativistic free particle. The conical tube was first solved in [11] with some particular choice of boundary conditions, here we solve the equation in general.

If the channel is a cone, the cross-sectional area is  $A(x) = \pi(1 + \lambda x)^2$ , where  $\lambda$  is the slope of the cone, and if  $D(x) = D_0$  is a constant, the Fick-Jacobs equation (1) takes the form

$$\frac{\partial C(x, t)}{\partial t} = D_0 \left( \frac{\partial^2 C(x, t)}{\partial x^2} - \frac{2\lambda}{1 + \lambda x} \frac{\partial C(x, t)}{\partial x} + \frac{2\lambda^2}{(1 + \lambda x)^2} C(x, t) \right). \quad (25)$$

Moreover, one can show that taking  $\beta^2 = D_0$  and  $f(x) = \ln(1 + \lambda x)$  this equation is equivalent to the Schrödinger wave equation:

$$-\frac{\partial C(x, t)}{\partial t} = \hat{H}_4 C(x, t). \quad (26)$$

Indeed, lets notice that making the identifications

$$\frac{\partial f(x)}{\partial x} = \frac{\lambda}{1 + \lambda x}, \quad \frac{\partial^2 f(x)}{\partial x^2} = \frac{-\lambda^2}{(1 + \lambda x)^2}, \quad \left( \frac{\partial f(x)}{\partial x} \right)^2 = \frac{\lambda^2}{(1 + \lambda x)^2}, \quad (27)$$

the Hamiltonian  $\hat{H}_4$ , given by equation (10), can be written as follows

$$\begin{aligned} \hat{H}_4 &= D_0 \left( \hat{P}^2 + 2i \frac{\lambda}{1 + \lambda x} \hat{P} - \frac{2\lambda^2}{(1 + \lambda x)^2} \right) \\ &= D_0 \left( -\frac{\partial^2}{\partial x^2} + 2 \frac{\lambda}{1 + \lambda x} \frac{\partial}{\partial x} - \frac{2\lambda^2}{(1 + \lambda x)^2} \right). \end{aligned} \quad (28)$$

With expression (28), we can see that equation (26) is equivalent to (25). In section 2, we showed that equation (26) is equivalent to a nonrelativistic

free particle. Therefore, the Fick-Jacobs equation with constant diffusion coefficient in a conical channel is equivalent to the Schrödinger equation of a nonrelativistic free particle.

Using the results of Section 2, we can find the solution of the equation (25) to be

$$C(x, t) = e^{f(x)} e^{-D_0 \hat{P}^2} e^{-f(x)} C_0(x), \quad (29)$$

which satisfies the initial condition  $C(x, 0) = C_0(x)$ . Considering the explicit form of  $f(x)$  one obtain

$$C(x, t) = (1 + \lambda x) e^{-D_0 \hat{P}^2} \left( \frac{C_0(x)}{1 + \lambda x} \right). \quad (30)$$

Notice that if

$$C_0(x) = 1 + \lambda x, \quad (31)$$

then

$$C(x, t) = 1 + \lambda x. \quad (32)$$

Thus, the system does not evolve.

Here we presented the relation between Schrödinger and Fick-Jacobs equations for a special geometry of the channel. However, the form of the channel can be different. In the next section we explore different geometries for the channel shape when the diffusion coefficient is constant.

## 4 Fick-Jacobs equation with constant diffusion coefficient

Motivated by the resulting mapping of the previous section, let's consider the case of channel with shape of a surface of revolution and when the diffusion coefficient is constant. In this case the Fick-Jacobs equation is equivalent to a Schrödinger equation and we study three cases that can be solved exactly.



To do so let's consider a constant diffusion coefficient  $D(x) = D_0 = \text{const.}$ , such that the Fick-Jacobs equation (1) takes the form

$$\frac{\partial C(x, t)}{\partial t} = D_0 \left[ \frac{\partial^2}{\partial x^2} - \frac{\partial \ln A(x)}{\partial x} \frac{\partial}{\partial x} + \left( \frac{\partial \ln A(x)}{\partial x} \right)^2 - \frac{1}{A(x)} \frac{\partial^2 A(x)}{\partial x^2} \right] C(x, t). \quad (33)$$

In the same way as in the previous section, we can now find a function  $f(x)$  and an effective potential  $V(x)$ , such that, we can construct the corresponding wave equation. Identifying

$$f(x) = \frac{1}{2} \ln A(x), \quad V(x) = \frac{1}{2} \frac{1}{A(x)} \frac{\partial^2 A(x)}{\partial x^2} - \frac{1}{4} \left( \frac{\partial \ln A(x)}{\partial x} \right)^2, \quad (34)$$

the equation (33) can be rewritten as

$$-\frac{\partial C(x, t)}{\partial t} = D_0 \left( \hat{P}^2 + 2i \frac{\partial f(x)}{\partial x} \hat{P} + \frac{\partial^2 f(x)}{\partial x^2} - \left( \frac{\partial f(x)}{\partial x} \right)^2 + V(x) \right) C(x, t). \quad (35)$$

Using the results of Section 2 we can see that this equation is equivalent to

$$-\frac{\partial C(x, t)}{\partial t} = e^{f(x)} \hat{H} e^{-f(x)} C(x, t), \quad \hat{H} = D_0 (\hat{P}^2 + V(x)), \quad (36)$$

which in turn is equivalent to

$$-\frac{\partial (e^{-f(x)} C(x, t))}{\partial t} = \hat{H} e^{-f(x)} C(x, t). \quad (37)$$

The general solution of this equation is

$$e^{-f(x)} C(x, t) = e^{-\hat{H}t} e^{-f(x)} C_0(x), \quad (38)$$

where the initial condition is  $C(x, 0) = C_0(x)$ . Therefore, the general solution of the Fick-Jacobs equation with constant diffusion coefficient  $D(x) = D_0$  is

$$C(x, t) = (e^{f(x)} e^{-\hat{H}t} e^{-f(x)}) C_0(x), \quad (39)$$

namely

$$C(x, t) = \sqrt{A(x)} e^{-\hat{H}t} \left( \frac{C_0(x)}{\sqrt{A(x)}} \right). \quad (40)$$

We can see, that if

$$C_0(x) = \sqrt{A(x)}, \quad (41)$$

then

$$C(x, t) = \sqrt{A(x)}. \quad (42)$$

Thus, the system does not evolve.

Now consider the following cases:

#### 4.1 Throat like channel

If the channel where the diffusion takes place is shaped like a throat, the cross-sectional area can be taken as an exponential function

$$A(x) = e^{\alpha x + \beta}, \quad (43)$$

where  $\alpha$  and  $\beta$  are arbitrary constants. In this case the potential defined in (34) is

$$V(x) = \frac{\alpha^2}{4}. \quad (44)$$

Therefore, in this case the general solution is

$$C(x, t) = e^{-\frac{D_0 \alpha^2 t}{4}} e^{\frac{\alpha x}{2}} e^{-D_0 \hat{P}^2 t} e^{-\frac{\alpha x}{2}} C_0(x). \quad (45)$$

#### 4.2 Sinusoidal channel

We can also obtain the solution of a channel with a different geometry, which has been studied in the literature, we mean the sinusoidal channel [10], for which the cross section is

$$A(x) = B (\sin \gamma x)^2, \quad (46)$$

being  $B$  and  $\gamma$  constants. In this case the general potential in (34) is simply

$$V(x) = -\gamma^2. \quad (47)$$

Then the solution is

$$C(x, t) = e^{D_0 \gamma^2 t} \sin(\gamma x) e^{-D_0 \hat{P}^2 t} \frac{C_0(x)}{\sin(\gamma x)}. \quad (48)$$

### 4.3 Gaussian channel

If the area of the cross section of the channel where the diffusion takes place is shaped like a Gaussian, we have

$$A(x) = e^{ax^2+bx+c}, \quad (49)$$

where  $a, b$  and  $c$  are constants. The potential defined in (34) in this case turns to be

$$V(\zeta) = a^2\zeta^2 + a, \quad \text{with} \quad \zeta = x + \frac{b}{a}. \quad (50)$$

Hence, the Hamiltonian takes the well know form

$$\hat{H} = D_0 \left( \hat{P}^2 + a^2\zeta^2 + a \right), \quad (51)$$

that is the Hamiltonian of an harmonic oscillator, as long as we identify the mass  $m = 1/(2D_0)$  and the frequency  $\omega^2 = 4D_0^2a^2$ . In this case, the general solution is

$$C(x, t) = e^{-D_0at} e^{\frac{a\zeta^2}{2}} e^{-D_0(\hat{P}^2+a^2\zeta^2)t} e^{-\frac{a\zeta^2}{2}} C_0(\zeta). \quad (52)$$

## 5 General Fick-Jacobs equation

In this section we study the general version of the Fick-Jacobs equation, we use a change of variable for which this equation can be written as a Schrödinger equation in a effective potential.

The Fick-Jacobs equation (1) can be written as

$$\begin{aligned} \frac{\partial C(x, t)}{\partial t} &= \frac{\partial}{\partial x} \left[ D(x) A(x) \frac{\partial}{\partial x} \left( \frac{C(x, t)}{A(x)} \right) \right] \\ &= D(x) \frac{\partial^2 C(x, t)}{\partial x^2} + D(x) \frac{\partial}{\partial x} \left[ \ln \left( \frac{D(x)}{A(x)} \right) \right] \frac{\partial C(x, t)}{\partial x} \\ &\quad - \frac{\partial}{\partial x} \left( D(x) \frac{\partial \ln A(x)}{\partial x} \right) C(x, t). \end{aligned} \quad (53)$$

Lets introduce the following change of variable

$$y = \int_{x_0}^x \frac{dz}{\sqrt{D(z)}}, \quad x_0 = \text{const.}, \quad (54)$$

notice that with this change of variable we have

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{D(x)}} \frac{\partial}{\partial y}, \quad D(x) \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial y^2} - \frac{\partial \left( \sqrt{D(x)} \right)}{\partial x} \frac{\partial}{\partial y}. \quad (55)$$

Then (53) takes the form

$$\begin{aligned} \frac{\partial C(y, t)}{\partial t} &= \frac{\partial^2 C(y, t)}{\partial y^2} + \sqrt{D(x)} \frac{\partial}{\partial x} \left( \ln \left[ \frac{\sqrt{D(x)}}{A(x)} \right] \right) \frac{\partial C(y, t)}{\partial y} \\ &\quad - \frac{\partial}{\partial x} \left( D(x) \frac{\partial \ln A(x)}{\partial x} \right) C(y, t). \end{aligned} \quad (56)$$

If the change of variable (54) is invertible, each of the terms of this equation can be written as a function of the new variable  $y$ . In this case we can define the next functions:

$$\begin{aligned} \frac{\partial}{\partial y} f(y) &= -\frac{1}{2} \sqrt{D(x)} \frac{\partial}{\partial x} \left[ \ln \left( \frac{\sqrt{D(x)}}{A(x)} \right) \right], \\ V(y) &= \left( \frac{\partial f(y)}{\partial y} \right)^2 - \frac{\partial^2 f(y)}{\partial y^2} + \frac{\partial}{\partial x} \left( D(x) \frac{\partial \ln A(x)}{\partial x} \right), \end{aligned} \quad (57)$$

then (56) can be written as

$$\frac{\partial C(y, t)}{\partial t} = \left( \frac{\partial^2}{\partial y^2} - 2 \frac{\partial f(y)}{\partial y} \frac{\partial}{\partial y} - \frac{\partial^2 f(y)}{\partial y^2} + \left( \frac{\partial f(y)}{\partial y} \right)^2 - V(y) \right) C(y, t).$$

Now, with the definitions

$$\hat{H}_f = \hat{P}^2 + 2i \frac{\partial f(y)}{\partial y} \hat{P} + \frac{\partial^2 f(y)}{\partial y^2} - \left( \frac{\partial f(y)}{\partial y} \right)^2 + V(y), \quad \hat{P} = -i \frac{\partial}{\partial y}, \quad (58)$$

we find that  $C(x, t)$  must satisfy

$$-\frac{\partial C(y, t)}{\partial t} = \hat{H}_f C(y, t). \quad (59)$$

Moreover, using the results of Section 2, we obtain

$$\hat{H}_f = e^{f(y)} \hat{H} e^{-f(y)}, \quad \hat{H} = \hat{P}^2 + V(y). \quad (60)$$

Therefore, the equation (59) can be written as

$$-\frac{\partial C(y, t)}{\partial t} = e^{f(y)} \hat{H} e^{-f(y)} C(y, t), \quad (61)$$

from where

$$-\frac{\partial \left( e^{-f(y)} C(y, t) \right)}{\partial t} = \hat{H} \left( e^{-f(y)} C(y, t) \right), \quad (62)$$

the solution of this equation is

$$e^{-f(y)} C(y, t) = e^{-\hat{H}t} \left( e^{-f(y)} C_0(y) \right). \quad (63)$$

Therefore, if the change of variables (54) is invertible, the general solution of the Fick-Jacobs equation is

$$C(y, t) = \left( e^{f(y)} e^{-\hat{H}t} e^{-f(y)} \right) C_0(y), \quad (64)$$

which satisfies the initial condition  $C(y, 0) = C_0(y)$ .

The results of this section are valid only if the change of variable (54) is invertible. There are several examples where this change of variable is invertible, for instance  $D(x) = x^n$ ,  $n = 1, 2, \dots$ , or  $D(x) = e^{\alpha x}$ ,  $\alpha = \text{const}$ .

In some interesting cases [10, 13], general diffusion coefficient is of the form

$$D(x) = \frac{D_0}{\left( 1 + \left[ \frac{d}{dx} \left( \sqrt{\frac{A(x)}{\pi}} \right) \right]^2 \right)^{1/2}}. \quad (65)$$

In that case depends only on the shape of the channel if the change of variable (54) is invertible or not. For instance, consider the case of the conical channel with cross section  $A(x) = \pi(1 + \lambda x)^2$ , the corresponding change of variable is

$$y = x \left( \frac{\sqrt{1 + \lambda^2}}{D_0} \right)^{\frac{1}{2}},$$

which is clearly invertible. We will need to take into account the invertibility of the change of variable, case by case, for (65) it only depends on the corresponding cross section area.

## 6 Summary

We have provided a connection between Fick-Jacobs diffusion equation and the Schrödinger equation. The Fick-Jacobs equation is a one dimensional effective equation with variable diffusion coefficient that describes the diffusion in a channel with a shape of a surface of revolution. To do so first we introduced a generalized momentum operator with which one can construct four different Hamiltonians, two of them are Hermitian and the other two are non-Hermitian. In the case of constant diffusion coefficient we obtained the following results: the case of the conical channel was mapped to quantum free particle; in the case when the shape of channel is a throat, is obtained a relation with quantum particle in a constant potential; also the sinusoidal channel was related with quantum particle in a constant potential; in addition when the channel cross section varies as a quadratic exponential, we obtained a relation with the quantum harmonic oscillator. The general case of variable diffusion coefficient was also considered. In this case we proposed a change of variable, if it is invertible, the Fick-Jacobs equation is equivalent to the Schrödinger equation of a particle in a effective potential.

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